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Some Classes of p-valent Analytic Functions Associated with Hypergeometric Functions

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Abstract. We define a linear operator on the class $\mathcal{A}(p)$ of p-valent analytic functions in the open unit disc involving Gauss hypergeometric functions and introduce certain new subclasses of $\mathcal{A}(p)$ using this operator. Some inclusion results, a radius problem and several other interesting properties of these classes are studied.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{m+p} z^{m+p}, \quad (p \in N = \{1, 2, 3, \dots\})$$

which are analytic and p-valent in the open unit disc $E = \{z : |z| < 1\}$. For

$$f(z) = \sum_{m=0}^{\infty} a_m z^m, \quad g(z) = \sum_{m=0}^{\infty} b_m z^m,$$

the Hadamard product (or convolution) is defined by

$$(f \star g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m$$

For $a \in \mathcal{R}, \quad c \in \mathcal{R} \backslash \mathcal{Z}_r^-$, where $\mathcal{Z}_r^- = \{\dots, -2, -1, 0\}$, define $L_p(a, c) : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ as

$$L_p(a, c) f(z) = \phi_p(a, c; z) \star f(z), \quad z \in E, f \in \mathcal{A}(p),$$

where

$$\phi_p(a, c; z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} z^{m+p}, \quad z \in E$$

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and $(\lambda)_\nu$ denotes the Pochhamer symbol (or the shifted factorial) defined(for $x, \nu \in C$ and in terms of the Gamma function) by

$$(x)_\nu = \frac{\Gamma(x+\nu)}{\Gamma(x)} = \begin{cases} 1, & (\nu = 0; \quad x \in C\backslash\{0\}), \\ x(x+1)\ldots(x+n-1), & (\nu = n \in N, x \in C). \end{cases}$$

The operator $L_p(a,c)$ was introduced by Saitoh [18]. This operator is an extension of Carlson-Shaffer operator $L_1(a,c)$, see [2].

For real or complex numbers a, b, c other than $0, -1, -2, \ldots$ the hypergeometric series is defined by

$$_2F_1(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \tag{1.1}$$

We note that series (1.1) converges absolutely for all $z \in E$ so that it represents an analytic function in E. Also

$$\phi_p(a,c;z)z^p = {}_2F_1(1,a,c;z).$$

We now introduce a function $(z^p{}_2F_1(a,b,c;z))^{(-1)}$ given by

$$(z^p{}_2F_1(a,b,c;z)) \star (z^p{}_2F_1(a,b,c;z))^{(-1)} = \frac{z^p}{(1-z)^{\lambda+p}}.$$

and obtain the following linear operator

$$I_\lambda(a,b,c)f(z) = (z^p{}_2F_1(a,b,c;z))^{(-1)} \star f(z), \tag{1.2}$$

where a, b, c are real other than $0, -1, -2, \ldots, \quad \lambda > -p, \quad z \in E$ and $f \in \mathcal{A}(p)$.

In particular, with $b = 1, p = 1$, I_λ was studied in [3] and for $a = n + p, b = c, \lambda = 1, p = 1$, see [15]. With some computation, we note that

$$I_\lambda(a,b,c)f(z) = z^p + \sum_{m=1}^{\infty} \frac{(c)_m(\lambda+p)_m}{(a)_m(b)_m} z^{m+p}. \tag{1.3}$$

From (1.2), it can easily be verified that

$$z\left(I_\lambda(a+1,b,c)f(z)\right)' = aI_\lambda(a,b,c)f(z) - (a-p)I_\lambda(a+1,b,c)f(z) \tag{1.4}$$

$$z\left(I_\lambda(a,b,c)f(z)\right)' = (\lambda+p)I_{\lambda+1}(a,b,c)f(z) - \lambda I_\lambda(a,b,c)f(z). \tag{1.5}$$

Let $P_k(\beta)$ be the class of functions $p(z)$ analytic in the unit disc E satisfying the properties $p(0) = 1$ and, for $z = re^{i\theta}, \quad k \geq 2$

$$\int_0^{2\pi} \left| Re\frac{p(z)-\beta}{(1-\beta)} \right| d\theta \leq k\pi, \quad (0 \leq \beta < 1). \tag{1.6}$$

For $\beta = 0$, we obtain the class P_k defined by Pinchuk [16] and for $k = 2, \beta = 0$, we have the class P of functions with positive real positive real part greater than β.

From (1.6), we can easily verify that $p \in P_k(\beta)$ if and only if there exists $p_1, p_2 \in P(\beta)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \quad z \in E. \tag{1.7}$$

We now define the following.

Definition 1.1. Let $f \in \mathcal{A}(p), z \in E$. Then $f \in R_k^\lambda(a,b,c,\beta,p)$ if and only if, for $k \geq 2, \quad 0 \leq \beta < 1$,

$$\left\{ \frac{z\left(z^{1-p} I_\lambda(a,b,c)f\right)'}{z^{1-p} I_\lambda(a,b,c)f} \right\} \in P_k(\beta), \quad z \in E.$$

In particular, $R_k^\lambda(a,\lambda+1,a,\beta,1) = R_k(\beta)$ is the class of functions of bounded radius rotation of order β, see [8, 12]. Also $R_2^\lambda(a,\lambda+1,a,\beta,1) \equiv S^\star(\beta)$, the c.lass of starlike univalent functions of order β.

We can define the class $V_k^\lambda(a,b,c,\beta,p)$ as follows.

Definition 1.2. Let $f \in \mathcal{A}(p)$. Then, for $z \in E$,

$$f \in V_k^\lambda(a,b,c,\beta,p) \Longleftrightarrow \frac{zf'(z)}{p} \in R_k^\lambda(a,b,c,\beta,p).$$

We note that $V_k^\lambda(a,\lambda+1,a,0,1) = V_k$ is the class of functions with bounded boundary rotation, and $V_2^\lambda(a,\lambda+1,a,\beta,1) = C(\beta)$, the class of convex univalent functions of order β.

Definition 1.3. Let $f \in \mathcal{A}(p)$. Then, for $k \geq 0, \alpha \geq 0, 0 \leq \beta < 1, z \in E, \quad f \in M_k^\lambda(a,b,c,\beta,p,\alpha)$ if and only if

$$\left\{ (1-\alpha)\frac{z(z^{1-p}I_\lambda(a,b,c)f)'}{z^{1-p}I_\lambda(a,b,c)f} + \alpha \frac{(z(z^{1-p}I_\lambda(a,b,c)f)')'}{(z^{1-p}I_\lambda(a,b,c)f)'} \right\} \in P_k(\beta).$$

We note that, for $\alpha = 1$, we have the class $V_k^\lambda(a,b,c,\beta,p)$ and $\alpha = 0$ gives us the class $R_k^\lambda(a,b,c,\beta,p)$.

Also $M_2^\lambda(a,\lambda+1,a,0,1) \equiv M(\alpha)$ is the class of α-starlike univalent functions and $M_k^\lambda(a,\lambda+1,a,0,1,\alpha)$ consists entirely of functions of bounded Mocanu variation, see [7].

In the above definitions, we obtain several known subclasses of analytic and multivalent functions by choosing the suitable values of the parameters k, λ, a, b, c and α. We would like to emphasize that a significant and important meromorphic extension of the linear operator $I_\lambda(a,b,c)$, popularly known as the Liu-Srivastava operator has been introduced and studied in [9]. For related work, see [5,6] for the analogous Dziok-Srivastava operator.

In the recent years, several interesting subclasses of analytic functions have been introduced and investigated, see [1,4,12,13,14,15,21,22].

For the sake of simplicity, we shall write $I_\lambda(a)$ in place of $I_\lambda(a,b,c)$ unless required otherwise.

2. Preliminary Results

Lemma 2.1 ([17]). *Let $p(z)$ be an analytic function in E with $p(0) = 1$ and $Re\{p(z)\} > 0$, $z \in E$. Then, for $s > 0$ and $\nu \neq -1$ (complex),*

$$Re\left\{ p(z) + \frac{szp'(z)}{p(z)+\nu} \right\} > 0, \quad for \quad |z| < r_0,$$

where r_0 is given by

$$r_0 = \frac{|\nu+1|}{\sqrt{A + (A^2 - |\nu^2 - 1|^2)^{\frac{1}{2}}}}, \; A = 2(s+1)^2 + |\nu|^2 - 1.$$

This result is best possible.

Lemma 2.2. *Let* $\beta_0 > 0, \beta_0 + \gamma > 0$ *and* $\alpha_1 \in [\alpha_0, 1)$, *where* $\alpha_0 = Max\left\{\frac{\beta_0-\gamma-n}{2\beta_0}, \frac{-\gamma}{\beta_0}\right\}$, $n \in N$. *If*

$$\left\{p(z) + \frac{nzp'(z)}{\beta_0 p(z) + \gamma}\right\} \in P(\alpha_1),$$

then

$$Re\{p(z)\} \geq \left[\frac{(\beta_0 + \gamma)}{{}_2F_1(\frac{2\beta_0}{n}(1-\alpha_1), 1, \frac{\beta_0+\gamma+n}{n}; \frac{1}{2})\beta_0} - \frac{\gamma}{\beta_0}\right], \tag{2.1}$$

and the bound in (2.2) is sharp, extremal function being

$$p_n(z) = \frac{1}{\beta_0 g_n(z)} - \frac{\gamma}{\beta_0},$$

where

$$\begin{aligned} g_n(z) &= \frac{1}{n}\int_0^1 \left[\frac{1-z}{1-tz}\right]^{\frac{2\beta_0(1-\alpha_1)}{n}} t^{\frac{\beta_0+\gamma}{n}-1} dt \\ &= \left[{}_2F_1\left(\frac{2\beta_0(1-\alpha_1)}{n}, 1, \frac{\beta_0+\gamma+n}{n}; \frac{z}{1-z}\right)\right]\left(\frac{1}{\beta_0+\gamma}\right). \end{aligned}$$

The above Lemma is a slightly modified version of Theorem 3.3e in [11, p113].

3. Main Results

Theorem 3.1. *Let* $\alpha > 0, \lambda \geq 0$ *and* $\beta \in [\gamma_0, 1)$*with* $\gamma_0 = Max\left\{\frac{1-\alpha}{2}, 0\right\}$. *Then*

(*i*). $M_k^{\lambda+1}(a,b,c,\beta,p,\alpha) \subset R_k^{\lambda+1}(a,b,c,\beta_1,p)$

(*ii*). $R_k^{\lambda+1}(a,b,c,\beta_1,p) \subset R_k^{\lambda}(a,b,c,\beta_2,p)$

(*iii*). $R_k^{\lambda}(a,b,c,\beta_2,p) \subset R_k^{\lambda}(a+1,b,c,\beta_3,p)$,

where

$$\beta_1 = \frac{1}{{}_2F_1(\frac{2}{\alpha}(1-\beta), 1, 1+\frac{1}{\alpha}; \frac{1}{2})} \tag{3.1}$$

$$\beta_2 = \frac{1+\lambda}{{}_2F_1(2(1-\beta_1), 1, 2+\lambda; \frac{1}{2})} \tag{3.2}$$

$$\beta_3 = \frac{1+a}{{}_2F_1(1(1-\beta_2), 1, 2+a; \frac{1}{2})} \tag{3.3}$$

The values β_i, $i = 1, 2, 3$ *are best possible.*

Proof. **(i).** set

$$\frac{z\left(z^{1-p}I_{\lambda+1}(a)f(z)\right)'}{z^{1-p}I_{\lambda+1}(a)f(z)} = H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z). \tag{3.4}$$

We note that H is analytic in E with $H(0) = 1$ and $H(z) \neq 0$ for all $z \in E$.
Since $f \in M_k^{\lambda+1}(a,b,c,\beta,p,\alpha)$, we have

$$\left[H(z) + \frac{\alpha z H'(z)}{H(z)}\right] \in P_k(\beta), \quad z \in E. \tag{3.5}$$

Using (3.4) with convolution techniques, it follows from (3.5) that

$$\left[H_i(z) + \frac{\alpha z H_i'(z)}{h_i(z)}\right] \in P(\beta), \quad z \in E, \quad i = 1, 2.$$

We now use Lemma 2.2 with $\gamma = 0, \beta_0 = \frac{1}{\alpha}, \alpha_1 = \beta, n = 1$ to have

$$ReH_i(z) \geq \frac{1}{{}_2F_1(\frac{2}{\alpha}(1-\beta), 1, 1+\frac{1}{\alpha}; \frac{1}{2})}, \quad i = 1, 2.$$

and this bound is sharp. Consequently $H \in P_k(\beta)$ and $f \in R_k^{\lambda+1}(a, b, c, \beta_1, p)$. This proves (i).

(ii). We now prove (ii). Let $f \in R_k^{\lambda+1}(a, b, c, \beta_1, p)$ and set

$$\frac{z\left(z^{1-p} I_\lambda(a) f(z)\right)'}{z^{1-p} I_\lambda(a) f(z)} = h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \tag{3.6}$$

where $h(z)$is analytic in E with $h(0) = 1$ and $h(z) \neq 0$ for all $z \in E$.

From (1.5) and (3.6), we have

$$\frac{z\left(z^{1-p} I_{\lambda+1}(a) f(z)\right)'}{z^{1-p} I_{\lambda+1}(a) f(z)} = \left\{h(z) + \frac{zh'(z)}{h(z) + \lambda}\right\} \in P_k(\beta_1),$$

where β_1 is given by (3.1).

Define

$$\phi_\lambda(z) = \frac{1}{\lambda+1} \frac{z^p}{1-z} + \frac{\lambda}{\lambda+1} \frac{z^p}{(1-z)^2}.$$

Then

$$\begin{aligned}
\left(h(z) + \frac{\phi_\lambda(z)}{z^p}\right) &= h(z) + \frac{zh'(z)}{h(z)+\lambda} \\
&= \left(\frac{k}{4} + \frac{1}{2}\right)\left[h_1(z) + \frac{zh_1'(z)}{h_1(z)+\lambda}\right] \\
&\quad -\left(\frac{k}{4} - \frac{1}{2}\right)\left[h_2(z) + \frac{zh_2'(z)}{h_2(z)+\lambda}\right].
\end{aligned}$$

Therefore it follows that

$$\left\{h_i(z) + \frac{zh_i'(z)}{h_i(z)+\lambda}\right\} \in P(\beta_1), \quad i = 1, 2, \quad z \in E.$$

Now using Lemma 2.2. with $n = 1, \beta_0 = 1, \quad \gamma = \lambda$, we have, for $i = 1, 2 \quad h_i \in P(\beta_2)$, where the exact value of β_2 is given by (3.2). This proves part (ii) of Theorem 3.1. Te last part (iii) of this inclusion result can easily be proved by using (1.4) and similar technique used above. □

Special Cases

(i). With $\alpha = 1, \quad f \in M_k^{\lambda+1}(a, b, c, \beta, p, 1) \equiv V_k^{\lambda+1}(a, b, c, \beta, p)$ and this implies $f \in R_k^{\lambda+1}(a, b, c, \beta_1, p)$, where $\beta_1 \in [0, 1)$, and

$$\beta_1 = \begin{cases} \frac{2\beta-1}{2-2^{2(1-\beta)}}, & \text{if} \quad \beta \neq \frac{1}{2} \\ \frac{1}{2ln2}, & \text{if} \quad \beta = \frac{1}{2} \end{cases} \tag{3.7}$$

(ii). For $\alpha = 1, b = \lambda + 1, c = a, p = 1$ and $k = 2$, we obtain a well-known result that a convex function of order β is starlike of order β_1, where β_1 is given by (3.7).
(iii). With $b = \lambda + 1, c = a, p = 1, \beta = 0, \alpha > 0$, we note that $f \in M_k\alpha)$, the class of functions of bounded Mocanu variation and by Theorem 3.1(i), it follows that f is in $R_k(\beta_1)$,
$\beta_1 = \frac{1}{{}_2F_1(\frac{2}{\alpha}, 1, 1+\frac{1}{\alpha}; \frac{1}{2})}$, for $z \in E$. The case $k = 2$ gives us the well-known result that α-starlike functions are starlike in E.

In brief, several interesting special cases can be obtained by choosing appropriate and suitable values of parameters a, b, c, p, k and λ.

Theorem 3.2. *Let $f \in R_k^\lambda(a, b, c, \beta, p)$ and define the integral operator $F_\mu(f$ by*

$$F_\mu(f)(z) = \frac{\mu + p}{z^\mu} \int_0^z t^{\mu-1} f(t)dt, \quad (\mu \geq 0). \tag{3.8}$$

Then $F_\mu \in R_k^\lambda(a, b, c, \delta, p)$ for $z \in E$, where exact values of δ is as given below.

$$\delta = \delta(\beta, \mu) = \left[\frac{(1 + \mu)}{{}_2F_1(2(1 - \beta), 1, 2 + \mu, \frac{1}{2})} - \mu\right]. \tag{3.9}$$

Proof. Let $f \in R_k^\lambda(a, b, c, \beta, p)$ and set

$$q(z) = \frac{z\left(z^{1-p} I_\lambda(a) F_\mu\right)'}{z^{1-p} I_\lambda(a) F_\mu},$$

where q is analytic in E with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in E$. With some computation, we obtain from (3.8),

$$\frac{z\left(z^{1-p} I_\lambda(a) f\right)'}{z^{1-p} I_\lambda(a) f} = \left[q(z) + \frac{zq'(z)}{q(z) + \mu}\right], \quad z \in E. \tag{3.10}$$

Using similar techniques as in the proof of previous Theorems, we see that $q \in P_k(\delta)$, where δ is given by (3.9).

Extremal function to show the sharpness is

$$\begin{aligned} q_1(z) &= \frac{1}{g_1(z)} - \mu = \int_0^1 \left[\frac{1 - z}{1 - tz}\right]^{2(1-\beta)} t^\mu dt \\ &= \left[{}_2F_1(2(1 - \beta), 1, 2 + \mu; \frac{z}{z - 1})\right](1 + \mu)^{-1}. \end{aligned} \tag{3.11}$$

□

In the following, we discuss the special case of (3.8) by choosing $\mu = 1$. We consider

$$F_1(f)(z) = \frac{p + 1}{z} \int_0^z f(t)dt. \tag{3.12}$$

For $p = 1$, this integral was discussed by Libera [10]. We have

Corollary 3.1. *Let $f \in R_k^\lambda(a, b, c, \beta, p)$, $\beta \in [\frac{-1}{2}, 1)$. Then F_1, defined by (3.12), belongs to $R_k^\lambda(a, b, c\delta_1, p)$ where δ_1 is given by (3.9) with $\mu = 1$.*

We note that $\delta_1(\frac{-1}{2}) = 0$ and $\delta(1) = 1$. Also, by choosing $p = 1$ and other parameters appropriately it can easily be seen that Libera integral operator maps starlike functions of order $(\frac{-1}{2})$ into starlike(univalent) functions. We shall now consider the converse case of Theorem 3.1.

Theorem 3.3. *Let $f \in R_k^{\lambda+1}(a,b,c,\beta,p)$. Then, for $\alpha > 0$, $f \in M_k^{\lambda+1}(a,b,c,\beta,p,\alpha)$ for $|z| < R_{\alpha,\beta}$, where $R_{\alpha,\beta}$ is given by (3.14) and this value is exact.*

Proof. Let

$$\frac{z\left(z^{1-p}I_{\lambda+1}(a)f(z)\right)'}{z^{1-p}I_{\lambda+1}(a)f(z)} = (1-\beta)H(z)+\beta, \quad z \in E,$$

where $H \in P_k$ and

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)q_2(z), \quad q_1, q_2 \in P, z \in E. \tag{3.13}$$

Proceeding as in Theorem 3.1, we have

$$\begin{aligned}\frac{1}{1-\beta}[(1-\alpha)\frac{z(z^{1-p}I_{\lambda+1}(a)f)'}{z^{1-p}I_{\lambda+1}(a)f} \quad &+ \quad \alpha\frac{(z(z^{1-p}I_{\lambda+1}(a)f)')'}{(z^{1-p}I_{\lambda+1}(a)f)'} - \beta] \\ &= \quad H(z) + \frac{\frac{\alpha}{1-\beta}zH'(z)}{H(z)+\frac{\beta}{1-\beta}} \\ &= \quad H(z) + \frac{\alpha_1 zH'(z)}{H(z)+\beta_1},\end{aligned}$$

where $\alpha_1 = \frac{\alpha}{1-\beta}$, $\beta_1 = \frac{\beta}{1-\beta}$.

Now define

$$\phi_{\alpha_1,\beta_1}(z) = \frac{1}{1+\beta_1}\frac{z^p}{(1-z)^{\alpha_1+1}} + \frac{\beta_1}{1+\beta_1}\frac{z^p}{(1-z)^{\alpha_1+2}}.$$

Then

$$\begin{aligned}\left(H \star \frac{\phi_{\alpha_1,\beta_1}(z)}{z^p}\right) &= H(z) + \frac{\alpha_1 zH'(z)}{H(z)+\beta_1} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[q_1(z) + \frac{\alpha_1 zq_1'(z)}{q_1(z)+\beta_1}\right] \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left[q_2(z) + \frac{\alpha_1 zq_2'(z)}{q_2(z)+\beta_1}\right], \quad q_i \in P, z \in E, \quad i = 1,2.\end{aligned}$$

We use Lemma 2.1, with $\nu = \frac{\beta}{1-\beta}, s = \frac{\alpha}{1-\beta} > 0$ to have

$$\left(q_i(z) + \frac{\alpha_1 zq_i'(z)}{q_i(z)+\beta_1}\right) \in P, \quad i = 1,2,$$

for

$$|z| < R_{\alpha,\beta} = \frac{|\nu+1|}{\sqrt{A + (A^2 - |\nu^2-1|^2)^{\frac{1}{2}}}}, \quad A = 2(s+1)^2 + |\nu|^2 - 1 \tag{3.14}$$

and this radius is exact. Consequently $f \in M_k^{\lambda+1}(a,b,c,\beta,p,\alpha)$ for $|z| < R_{\alpha,\beta}$ and the exact value of $R_{\alpha,\beta}$ is given by (3.14). □

As a special case, for $\beta = 0, \alpha = 1, \nu = 0, s = 1$ and $A = 7$, we have $R_{1,0} = \frac{1}{\sqrt{7+48}} \approx 0.268 \approx 2 - \sqrt{3}$.

Remark 3.4. *The radii for the converse cases of other parts of Theorem 3.1 and Theorem 3.2 can be obtained by the similar procedure and techniques applied in Theorem 3.3.*

Conclusion. In this paper, we have defined a linear operator on the class $\mathcal{A}(p)$ of p-valent analytic functions in the open unit disc involving Gauss hypergeometric functions. Using this linear operator, we have introduced and investigated certain new subclasses of $\mathcal{A}(p)$. Some inclusion results, a radius problem and several other interesting properties of these classes are studied. Results proved in this paper may stimulate further research activities in this dynamic field.

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